Solutions to open problems in Neeb's recent survey on infinite-dimensional Lie groups

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Abstract

We solve three open problems concerning infinite-dimensional Lie groups posed in a recent survey article by K.-H. Neeb [21]. Moreover, we prove a result by the author announced in [21], which answers a question posed in an earlier, unpublished version of the survey.

Introduction

The recent article [21] by K.-H. Neeb surveyed the state of the art of the theory of infinite-dimensional Lie groups, and described typical open problems. In this article, we provide full solutions to three of these:

- We describe a subgroup of an infinite-dimensional Lie group which is not an initial Lie subgroup (Section 1). This answers [21, Problem II.6], the "Initial Subgroup Problem," in the negative.

- We show that every subgroup of a direct limit $G = \varinjlim G_n$ of finite-dimensional Lie groups is an initial Lie subgroup (Section 2). This answers [21, Problem VII.3] in the affirmative.

- We show that every direct limit $G = \varinjlim G_n$ of finite-dimensional Lie groups is a topological group with Lie algebra in the sense of [16] (Section 3). This answers [21, Problem VII.2] in the affirmative.

We also prove a result by the author announced in [21, Proposition VI.2.11]: Let M be σ-compact finite-dimensional smooth manifold of dimension $n > 0$, and $\mathrm{Diff}_c(M)$ be the Lie group of smooth diffeomorphisms $\gamma \colon M \to M$ which are compactly supported in the sense that the closure of $\{x \in M \colon \gamma(x) \neq x\}$ is compact (see [18] and [9]). Given $p \in M$, let $\mathrm{Diff}_c(M)_p$ be its stabilizer, and $\mathrm{Diff}_c(M)_{p,0}$ be the subgroup of all $\gamma \in \mathrm{Diff}_c(M)_p$ such that $\det T_p(\gamma) > 0$, where $T_p(\gamma) \colon T_pM \to T_pM$ is the tangent map. Let $\mathrm{Gf}_n(\mathbb{R}) \subseteq \mathbb{R}[\![x_1, \ldots, x_n]\!]$ be the group of all formal diffeomorphisms of $\mathbb{R}^n$, and $\mathrm{Gf}_n(\mathbb{R})_0$ be the subgroup of all formal diffeomorphisms whose linear part has positive determinant (see [21, pp. 426–427] and [23]). In Section 4, we establish a version of É. Borel's theorem on the existence of smooth maps with a given Taylor series:

Borel's Theorem for Diffeomorphisms. *Let a finite-dimensional smooth manifold M of dimension n be given, $p \in M$ and ϕ be a local chart around p such that $\phi(p) = 0$. For each $\gamma \in \mathrm{Diff}_c(M)$, let $T_p^\infty(\gamma)$ denote the Taylor series of $\phi \circ \gamma \circ \phi^{-1}$ around 0. Then*

$$T_p^\infty \colon \mathrm{Diff}_c(M)_{p,0} \to \mathrm{Gf}_n(\mathbb{R})_0$$

is a surjective homomorphism of Lie groups.

We mention that further problems from [21] have an affirmative answer:

In [3], it is shown that the group $\mathrm{Gh}_n(\mathbb{C})$ of germs of analytic diffeomorphisms around $0 \in \mathbb{C}^n$ (as first considered in [22]) is a regular Lie group in Milnor's sense.[1] This answers the first half of [21, Problem VI.5] in the affirmative.

Using a version of the Frobenius Theorem for co-Banach distributions on infinite-dimensional manifolds,[2] one obtains the following result concerning integral subgroups in the sense of [21, Definition IV.4.7] (see [11]):

Let G be a Lie group modelled on a locally convex space and $\mathfrak{h} \subseteq L(G)$ be a Lie subalgebra such that $L(G) = \mathfrak{h} \oplus X$ for some Banach space $X \subseteq L(G)$. Then there exists an integral subgroup $H \subseteq G$ such that $L(H) = \mathfrak{h}$.

Note that a complement X always exists if $\mathfrak{h}$ has finite codimension. Hence [21, Problem VI.4] has an affirmative answer (even if G is not regular).

Finally, we mention that the idea expressed in [21, Problem VII.5] has been taken up in [8, Lemma 14.5 and Section 15].

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[1]Likewise for groups of germs of diffeomorphisms around a compact set, as in [8].

[2]The adaptation to Keller's C_c^∞-theory (our setting of infinite-dimensional calculus) of Hiltunen's Frobenius Theorem from [13] (to be found in [11]).

Prerequisites. The reader is referred to [21], [12] or [4] for the setting of infinite-dimensional Lie groups, manifolds and C^k-mappings used in this article (cf. [19] for the case of sequentially complete modelling spaces). The words "Lie group" and "manifold" will always refer to Lie groups and manifolds modelled on locally convex spaces.

1 Example of a subgroup of a Lie group which is not an initial Lie subgroup

We shall use the following terminology.

Definition 1.1 Let $\mathbb{N} := \{1, 2, \ldots\}$, $k \in \mathbb{N} \cup \{\infty\}$ and M be a C^k-manifold. A subset $N \subseteq M$ is called a C^k-*initial submanifold* of M if N can be made a C^k-manifold with the following properties:

(a) The inclusion map $\lambda \colon N \to M$ is a C^k-map; and

(b) If X is a C^k-manifold and $f \colon X \to N$ a map such that $\lambda \circ f$ is C^k, then f is C^k.

Such a manifold structure is called C^k-*initial* in M. If G is a Lie group, let $L(G) := T_1(G)$ be its Lie algebra. Following [21, Definition II.6.1], we say that a subgroup $H \subseteq G$ is an *initial Lie subgroup* if it can be given a smooth manifold structure which makes H a Lie group, is C^k-initial in G for each $k \in \mathbb{N} \cup \{\infty\}$, and gives rise to an injective tangent map $L(\lambda) := T_1(\lambda) \colon L(H) \to L(G)$.

Remark 1.2 It means additional information if N and X may even be C^k-manifolds with C^k-boundary in Definition 1.1, or more generally C^k-manifolds "with rough boundary" as in [12].[3] We then call N a *strongly initial C^k-submanifold* of M. Notably, the strengthened concept enables maps $f \colon X \to N$ to be considered if $X = [0, 1]$, $X = [0, 1]^2$ or X is a simplex.

Let $\mathbb{R}^{\mathbb{N}}$ be the Fréchet space of real sequences (with the product topology). We show:

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[3]The latter manifolds locally look like a locally convex subset with dense interior in the modelling space. They include manifolds with C^k-boundary and manifolds with corners as special cases.

Theorem 1.3 *For each $k \in \mathbb{N} \cup \{\infty\}$, the additive subgroup $N := \ell^\infty$ of bounded sequences is not a C^k-initial submanifold of $M := \mathbb{R}^\mathbb{N}$. In particular, it is not an initial Lie subgroup.*

Proof. We shall use a well-known fact: A map from a C^k-manifold to $\mathbb{R}^\mathbb{N}$ (or any direct product of locally convex spaces) is C^k if and only if all of its components are C^k (see [12, Chapter 1] or [1, Lemma 10.2]).

It is useful to note first that, for each $m \in \mathbb{N}$, the topological product $K_m := [-m, m]^\mathbb{N}$ is a metrizable compact topological space (and therefore separable), whence every open subset $U \subseteq K_m$ is σ-compact.

If the theorem was false, we could equip N with a C^k-initial manifold structure. Assuming this, write $\lambda \colon N \to M$ for the inclusion map (which is C^k). Given $m \in \mathbb{N}$, choose an injective C^k-map $h_m \colon \mathbb{R} \to \,]{-}2m, 2m[$ such that $h_m([-1, 1]) = [-m, m]$. Then the injective map

$$g_m \colon \mathbb{R}^\mathbb{N} \to N, \quad (x_n)_{n \in \mathbb{N}} \mapsto (h_m(x_n))_{n \in \mathbb{N}}$$

is C^k by initiality, exploiting that $\lambda \circ g_m$ is C^k since its components are C^k. In particular, g_m is continuous, entailing that

$$g_m|_{[-1,1]^\mathbb{N}} \colon [-1, 1]^\mathbb{N} \to N$$

is a topological embedding. Since $g_m([-1, 1]^\mathbb{N}) = K_m$ as a set and $\lambda \circ g_m$ takes $[-1, 1]^\mathbb{N}$ homeomorphically onto K_m, it follows that N induces the given compact topology on K_m.

By the preceding, $N = \bigcup_{m \in \mathbb{N}} K_m$ is σ-compact. Let E be the modelling space of N and $\phi \colon N \supseteq U \to V \subseteq E$ be a chart of N, defined on a non-empty open set $U \subseteq N$. Then $U = \bigcup_{m \in \mathbb{N}} (U \cap K_m)$ where $U \cap K_m$ is σ-compact for each $m \in \mathbb{N}$ (as recalled above), and hence U is σ-compact. Consequently, also the locally convex space E is σ-compact, because it contains a σ-compact set with non-empty interior.

If $x \in N$, then $T_x(N) \cong E$ is σ-compact. We claim that the tangent map $T_x(\lambda) \colon T_x(N) \to T_x(M)$ is surjective.

If this is true, then $\mathbb{R}^\mathbb{N} \cong T_x(M) = T_x(\lambda)(T_x(N))$ is σ-compact. But this is absurd (by a Baire argument), and the theorem is proved by contradiction. To prove the claim, identify $T_x(M)$ with $\mathbb{R}^\mathbb{N}$. If $y = (y_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}$, then

$$g \colon \mathbb{R} \to N, \quad g(t) := (x_n + \sin(ty_n))_{n \in \mathbb{N}}$$

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is a function such that $\lambda \circ g \colon \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ is C^k, because all components of this composition are C^k. Furthermore, $T_x(\lambda)(g'(0)) = (\lambda \circ g)'(0) = y$. $\qquad \square$

Our proof shows that Theorem 1.3 remains valid if, instead of our general setting, only manifolds modelled on complete (or sequentially complete, or Mackey complete) locally convex spaces are considered. Also, it remains valid if manifolds are replaced by manifolds with rough boundary.

2   Subgroups of direct limit groups are initial

If $G_1 \subseteq G_2 \subseteq \cdots$ is a sequence of finite-dimensional real Lie groups such that the inclusion maps $G_n \to G_{n+1}$ are smooth homomorphisms, then $G := \bigcup_{n \in \mathbb{N}} G_n$ admits a Lie group structure making each inclusion map $G_n \to G$ a smooth homomorphism, and such that $G = \varinjlim G_n$ in the category of smooth Lie groups modelled on locally convex spaces [7, Theorem 4.3] (cf. [20], [17, Theorem 47.9] and [5] for special cases). Then $G = \varinjlim G_n$ also as a topological space, and as a C^k-manifold, for each $k \in \mathbb{N}_0 \cup \{\infty\}$. Furthermore,

$$L(G) \;=\; \bigcup_{n \in \mathbb{N}} L(G_n) \;=\; \varinjlim L(G_n)$$

as a locally convex space, topological space and C^k-manifold (see [7, Theorems 4.3 (a) and 3.1] for all of this). For brevity, we shall refer to $G = \varinjlim G_n$ as a *direct limit Lie group.* In this section, we show:

Theorem 2.1 *Let* $G_1 \subseteq G_2 \subseteq \cdots$ *be finite-dimensional Lie groups, such that the inclusion map* $G_n \to G_{n+1}$ *is a smooth homomorphism for each* $n \in \mathbb{N}$. *Let* $G = \varinjlim G_n$ *be the direct limit Lie group, and* $H \subseteq G$ *be a subgroup. Equip* $H_n := G_n \cap H$ *with the finite-dimensional real Lie group structure induced by* G_n *(as in* [2, Chapter III, §4.5, Proposition 9]*) and* H *with the Lie group structure making it the direct limit Lie group* $H = \varinjlim H_n$ *(as in* [7, Proposition 7.2]*). Then the manifold structure on* H *is strongly C^k-initial in* G, *for all* $k \in \mathbb{N} \cup \{\infty\}$. *Thus* H *is an initial Lie subgroup of* G.

The final conclusion is possible because the inclusion map $\lambda \colon H \to G$ gives rise to an embedding $L(\lambda) \colon L(H) \to L(G)$ by [7, Proposition 7.2].

It is useful to recall some simple facts.

2.2 The identity component of a direct limit Lie group $G = \varinjlim G_n$ is

$$G_0 \;=\; \bigcup_{n \in \mathbb{N}} (G_n)_0 \;=\; \varinjlim (G_n)_0 \,.$$

In particular, G_0 is a direct limit group as well.

2.3 Consider an ascending sequence $X_1 \subseteq X_2 \subseteq \cdots$ of topological spaces. Then $X := \bigcup_{n \in \mathbb{N}} X_n$ is its direct limit in the category of topological spaces, if we declare $U \subseteq X$ open if and only if $U \cap X_n$ is open in X_n for each $n \in \mathbb{N}$. The topology so obtained is called the *direct limit topology*. If $U_n \subseteq X_n$ are open subsets such that $U_1 \subseteq U_2 \subseteq \cdots$, then $U := \bigcup_{n \in \mathbb{N}} U_n$ is open in X, because $U \cap X_n = \bigcup_{m \geq n} U_m \cap X_n$ is open in X_n for each n.

Also the following lemma will be useful for the proof of Theorem 2.1.

Lemma 2.4 *Let X be a finite-dimensional real vector space, $K \subseteq X$ be a non-empty compact set, $k \in \mathbb{N} \cup \{\infty\}$, M be a finite-dimensional C^k-manifold and $f \colon U \to M$ be a C^k-map on an open neighbourhood U of K in X such that $f|_K$ is injective and the tangent map $T_x(f) \colon T_x(X) \to T_{f(x)}(M)$ is invertible for each $x \in K$. Then there is an open neighbourhood $V \subseteq U$ of K such that $f(V)$ is open in M and $f|_V$ is a C^k-diffeomorphism onto $f(V)$.*

Proof. The proof of [8, Lemma 15.6] (where $M = X$) carries over.[4] $\qquad\square$

Proof of Theorem 2.1. We may assume that $G_1 = H_1 = \{1\}$. Let X be a C^k-manifold (possibly with rough boundary) and $f \colon X \to H$ be a map such that $\lambda \circ f \colon X \to G$ is C^k, where $\lambda \colon H \to G$ is the inclusion map. Given $p \in X$, let Y be the connected component of X which contains p. Since Y is open in X, it suffices to show that $f(p)^{-1} f|_Y$ is C^k on some open neighbourhood of p. After replacing X with Y and f with $f(p)^{-1} f|_Y$, we may assume that X is connected and $f(p) = 1$. Then $f(X)$ is contained in the identity component G_0 of G. Since $G_0 = \varinjlim (G_n)_0$ (see **2.2**), after replacing G with G_0, G_n with $(G_n)_0$ and H with $\overrightarrow{H} \cap G_0$, we may assume that G and each G_n is connected. Then G_n is the integral subgroup of G_{n+1} with Lie algebra $L(G_n)$. Since the latter is C^∞-initial in G_{n+1}, it follows that

$$L(G_n) \;=\; \{x \in L(G_{n+1}) \colon \exp_{G_{n+1}}(\mathbb{R}x) \subseteq G_n\} \tag{1}$$

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[4]Unfortunately, there is a misprint: "E" reads "X" in the lemma.

(see [21, Theorem IV.4.14]). We recall that

$$L(H_n) = \{\gamma'(0)\colon \gamma \in C^1([0,1], G_n) \text{ with } \gamma(0) = 1 \text{ and } \gamma([0,1]) \subseteq H_n\} \quad (2)$$

and also

$$L(H_n) = \{x \in L(G_n)\colon \exp_{G_n}(\mathbb{R}x) \subseteq H_n\}, \quad (3)$$

(cf. [2, Chapter 3, §4.5] or [21, Theorem IV.4.14]). Combining (3) with (1) and the fact that $L(G) = \bigcup_{n \in \mathbb{N}} L(G_n)$, we see that

$$L(H_n) = \{x \in L(G)\colon \exp_G(\mathbb{R}x) \subseteq H_n\}$$

and thus

$$L(H) \cap L(G_n) = L(H_n). \quad (4)$$

Recursively, we find vector subspaces $E_n \subseteq L(G_n)$ for $n \in \mathbb{N}$ such that $E_{n-1} \subseteq E_n$ (if $n \geq 2$) and

$$L(G_n) = L(H_n) \oplus E_n.$$

In fact, if $n = 1$ we choose $E_1 := \{0\}$. If $E_1, \ldots, E_n$ have been chosen, then

$$L(H_{n+1}) \cap E_n = L(H_{n+1}) \cap L(G_n) \cap E_n = L(H_n) \cap E_n = \{0\},$$

using (4) for the penultimate equality. Hence E_n can be extended to a vector complement E_{n+1} to $L(H_{n+1})$ in $L(G_{n+1})$.

Give $E := \bigcup_{n \in \mathbb{N}} E_n = \varinjlim E_n$ the locally convex direct limit topology. Then

$$L(G) = \varinjlim (L(H_n) \oplus E_n) = \varinjlim L(H_n) \oplus \varinjlim E_n = L(H) \oplus E$$

as a locally convex space (see, e.g., [14, Theorem 3.4]). For each $n \in \mathbb{N}$, let $R_n \subseteq L(H_{n+1})$ and $S_n \subseteq E_{n+1}$ be vector subspaces such that

$$L(H_{n+1}) = L(H_n) \oplus R_n \quad \text{and} \quad E_{n+1} = E_n \oplus S_n.$$

We now pick a basis $\mathcal{B}$ of $L(G)$ such that $\mathcal{B} \cap L(H_n)$ is a basis of $L(H_n)$, $\mathcal{B} \cap E_n$ is a basis of E_n, $\mathcal{B} \cap R_n$ is a basis of R_n and $\mathcal{B} \cap S_n$ is a basis of S_n, for each $n \in \mathbb{N}$. It can be used to introduce the supremum norm

$$\|.\|\colon L(G) \to [0, \infty[, \quad \left\| \sum_{b \in \mathcal{B}} t_b b \right\| := \sup_{b \in \mathcal{B}} |t_b| \quad \text{for } (t_b)_{b \in \mathcal{B}} \in \mathbb{R}^{(\mathcal{B})}.$$

Given $t > 0$ and $n \in \mathbb{N}$, abbreviate

$$U_t^n := \{x \in L(H_n) \colon \|x\| < t\}, \qquad V_t^n := \{x \in E_n \colon \|x\| < t\},$$

$$X_t^n := \{x \in R_n \colon \|x\| < t\}, \quad \text{and} \quad Y_t^n := \{x \in S_n \colon \|x\| < t\}.$$

Then $U_t^{n+1} = U_t^n \times X_t^n$ and $V_t^{n+1} = V_t^n \times Y_t^n$. Let $2 = t_1 > t_2 > \cdots > 1$.

Claim. *There are C^∞-maps $\eta_n \colon V_{t_n}^n \to G_n$ for $n \in \mathbb{N}$ and C^∞-diffeomorphisms $\gamma_n \colon U_{t_n}^n \to H_n$ onto open identity neighbourhoods in H_n, such that $\gamma_n(0) = \eta_n(0) = 1$,*

$$g_n \colon U_{t_n}^n \times V_{t_n}^n \to G_n, \quad g_n(x, y) := \gamma_n(x)\eta_n(y)$$

is a C^∞-diffeomorphism onto an open identity neighbourhood in G_n, and

$$\gamma_k|_{U_{t_k}^\ell} = \gamma_\ell|_{U_{t_k}^\ell} \quad \text{as well as} \quad \eta_k|_{V_{t_k}^\ell} = \gamma_\ell|_{V_{t_k}^\ell} \quad \text{for all } 1 \le \ell \le k \le n. \tag{5}$$

If this claim is true, then $\sigma_n := \gamma_n|_{U_1^n}$ is a C^∞-diffeomorphism onto the open identity neighbourhood $Q_n := \gamma_n(U_1^n)$ in H_n, the mapping $h_n := g_n|_{U_1^n \times V_1^n}$ is a C^∞-diffeomorphism onto the open identity neighbourhood $g_n(U_1^n \times V_1^n) =: P_n$ in G_n, and $\tau_n := \eta_n|_{V_1^n} \colon V_1^n \to G_n$ is a smooth map (and in fact an immersion and topological embedding, since $h_n(x, y) = \sigma(x)\tau(y)$ and h_n is a diffeomorphism.). Then $U := \bigcup_{n \in \mathbb{N}} U_1^n$ is an open 0-neighbourhood in $L(H)$ and $Q := \bigcup_{n \in \mathbb{N}} Q_n$ is a connected, open identity neighbourhood in H (see **2.3**). Moreover,

$$\sigma := \varinjlim \sigma_n \colon L(H) \supseteq U \to Q \subseteq H$$

is a C^∞-diffeomorphism (cf. [7, Lemma 1.9 and Proposition 3.3]). Also, $W := \bigcup_{n \in \mathbb{N}} (U_1^n \times V_1^n)$ is an open 0-neighbourhood in $L(G)$, $P := \bigcup_{n \in \mathbb{N}} P_n$ is an open identity neighbourhood in G, and

$$h := \varinjlim h_n \colon L(G) \supseteq W \to P \subseteq G$$

is a C^∞-diffeomorphism. Finally, $V := \bigcup_{n \in \mathbb{N}} V_n$ is an open 0-neighbourhood in E and

$$\tau := \varinjlim \tau_n \colon E \supseteq V \to G$$

is a C^∞-map (and actually an immersion and topological embedding, since $h(x, y) = \sigma(x)\tau(y)$ and h is a diffeomorphism). There exists a connected open neighbourhood $Z \subseteq X$ of p such that $f(Z) \subseteq P$. We now show that

$$f(Z) \subseteq Q. \tag{6}$$

To this end, let $q \in Z$. There exists a C^1-map $\theta \colon [0,1] \to Z$ such that $\theta(0) = p$ and $\theta(1) = q$. By [7, Lemma 1.7 (d)], there exists $n \in \mathbb{N}$ such that

$$f(\theta([0,1])) \subseteq G_n \,.$$

Then the left logarithmic derivative $\delta(f \circ \theta) \colon [0,1] \to L(G)$ is a continuous map taking its values in $L(H_n)$ (by (2) and [2, Chapter 3, §4.5, Lemma 4]).[5] As a consequence of the standard Existence and Uniqueness Theorem for solutions to ordinary differential equations, there exists a unique C^1-curve $\kappa \colon [0,1] \to (H_n)_0$ such that $\delta(\kappa) = \delta(f \circ \theta)$ and $\kappa(0) = 1$. Then $\kappa = f \circ \theta$ by uniqueness of product integrals (cf. [19, Lemma 7.4]), entailing that $f(\theta([0,1])) = \kappa([0,1])$ is a compact subset of $(H_n)_0$. Note that if $y \in V$, then $Q\tau(y) \cap H_0 \neq \emptyset$ if and only if $\tau(y) \in H_0$, in which case $Q\tau(y)$ is an open subset of H_0. Let

$$D := \{y \in V \colon Q\tau(y) \cap H_0 \neq \emptyset\} \,.$$

Then $(Q\sigma(y))_{y \in D}$ is an open cover of $f(\theta([0,1]))$ by disjoint open subsets of H_0. Since $f(\theta([0,1]))$ is connected, it follows that $f(\theta([0,1])) \subseteq Q\sigma(0) = Q$. Hence $f(q) = f(\theta(1)) \in Q$ in particular, establishing (6).

Because $P = Q\tau(V) \cong Q \times V$ and $Q \times \{0\}$ obviously is strongly C^k-initial in $Q \times V$, it follows that Q is strongly C^k-initial in P. Since $f(Z) \subseteq Q$ and $f|_Z$ is C^k as a map to P, we infer that $f|_Z \colon Z \to Q$ is C^k and hence also $f|_Z \colon Z \to H$.

It only remains to verify the claim. If $n = 1$, there is one (and only one) choice of $\gamma_1, \eta_1 \colon \{0\} \to \{1\}$. Now assume that $n \in \mathbb{N}$ and assume that $\gamma_1, \ldots, \gamma_n$ and $\eta_1, \ldots, \eta_n$ have already been constructed with the desired properties. Let $A \subseteq R_n$ be an open 0-neighbourhood and $\mu \colon A \to H_{n+1}$ be a smooth map such that $\mu(0) = 1$ and $T_0(\mu) = \mathrm{id}_{R_n}$, identifying $T_0(R_n)$ with R_n. Consider the map

$$\alpha \colon A \times U_{t_n}^n \times V_{t_n}^n \to G_{n+1}\,, \quad (a, x, y) \mapsto \mu(a) g_n(x, y) \,.$$

Then

$$\mathrm{im}(T_{(0,x,y)}(\alpha)) = \gamma_n(x) \cdot (R_n \oplus L(G_n)) \cdot \eta_n(y) \tag{7}$$

for all $x \in U_{t_n}^n$ and $y \in V_{t_n}^n$ (where $\cdot$ denotes multiplication in the tangent group TG_{n+1}). In fact, abbreviating $c := \gamma_n(x)$ and $e := \eta_n(y)$, the image is

$$
\begin{aligned}
R_n \cdot ce + \operatorname{im}\left(T_{(x,y)}g_n\right) &= R_n \cdot ce + T_{ce}G_n = R_n \cdot ce + c \cdot L(G_n) \cdot e \\
&= R_n \cdot ce + c \cdot L(H_n) \cdot e + c \cdot E_n \cdot e \\
&= R_n \cdot ce + L(H_n) \cdot ce + c \cdot E_n \cdot e \\
&= (R_n + L(H_n)) \cdot ce + c \cdot E_n \cdot e \\
&= L(H_{n+1}) \cdot ce + c \cdot E_n \cdot e \\
&= c \cdot L(H_{n+1}) \cdot e + c \cdot E_n \cdot e \\
&= c \cdot (L(H_{n+1}) + E_n) \cdot e \,,
\end{aligned}
$$

where $L(H_{n+1}) + E_n = R_n \oplus L(H_n) \oplus E_n = R_n \oplus L(G_n)$.

We now equip $L(G_{n+1})$ with an inner product, allowing us to consider orthogonal complements of vector subspaces of $L(G_{n+1})$. Setting

$$
K_y := \left(\eta_n(y)^{-1} \cdot (R_n \oplus L(G_n)) \cdot \eta_n(y)\right)^{\perp}
$$

for $y \in V_{t_n}^n$, we obtain a vector subbundle $K := \bigcup_{y \in V_{t_n}^n} \{y\} \times K_y$ of the trivial bundle $V_{t_n}^n \times L(G_{n+1}) \to V_{t_n}^n$. Because $V_{t_n}^n$ is smoothly contractible, this vector bundle is trivial as a smooth vector bundle (by the C^∞-version of [15, Chapter 2, Theorem 2.4]), and hence there exists a fibre-preserving C^∞-diffeomorphism

$$
\phi \colon V_{t_n}^n \times S_n \to K \,,
$$

where $V_{t_n}^n \times S_n \to V_{t_n}^n$ is the trivial bundle. Let $\pi_2 \colon V_{t_n}^n \times L(G_{n+1}) \to L(G_{n+1})$ be the projection onto the second factor and $\psi \colon C \to G_{n+1}$ be a C^∞-map, defined on an open 0-neighbourhood $C \subseteq L(G_{n+1})$, such that $\psi(0) = 1$ and $T_0\psi = \operatorname{id}_{L(G_{n+1})}$. Then $B := (\pi_2 \circ \phi)^{-1}(C)$ is an open neighbourhood of $V_{t_n}^n \times \{0\}$ in $V_{t_n}^n \times S_n$, and

$$
\nu \colon B \to G_{n+1} \,, \quad \nu(z) := (\psi \circ \pi_2 \circ \phi)(z)
$$

is a smooth map such that $\nu(y, 0) = 1$ for all $y \in V_{t_n}^n$ and

$$
\operatorname{im}(T_{(y,0)}\nu) = \operatorname{im} T_0\nu(y, \bullet) = K_y \,.
$$

Then the smooth map

$$
\zeta \colon U_{t_n}^n \times A \times B \to G_{n+1} \,, \quad (x, a, y, b) \mapsto \mu(a)\gamma_n(x)\eta_n(y)\nu(y, b)
$$

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(for $x \in U_{t_n}^n$, $a \in A$, $y \in V_{t_n}^n$ and $b \in S_n$ such that $(y, b) \in B$) satisfies

$$
\begin{aligned}
\mathrm{im}\,(T_{(x,0,y,0)}\zeta) &= \gamma_n(x) \cdot (R_n \oplus L(G_n)) \cdot \eta_n(y) + \gamma_n(x)\eta_n(y) \cdot K_y \\
&= \gamma_n(x)\eta_n(y) \cdot \big(\eta_n(y)^{-1} \cdot (R_n \oplus L(G_n)) \cdot \eta_n(y) \\
&\qquad + (\eta_n(y)^{-1} \cdot (R_n \oplus L(G_n)) \cdot \eta_n(y))^{\perp} \big) \\
&= \gamma_n(x)\eta_n(y) \cdot L(G_{n+1}) = T_{\zeta(x,0,y,0)}G_{n+1}\,,
\end{aligned}
$$

using (7) for the first equality. Hence $T_{(x,0,y,0)}\zeta$ is a linear isomorphism (by reasons of dimension). The restriction of ζ to $\overline{U_{t_{n+1}}^n} \times \{0\} \times \overline{V_{t_{n+1}}^n} \times \{0\}$ corresponds to the restriction of g_n to $\overline{U_{t_{n+1}}^n} \times \overline{V_{t_{n+1}}^n}$ and hence is injective. Thus Lemma 2.4 provides an open neighbourhood Ω of $\overline{U_{t_{n+1}}^n} \times \{0\} \times \overline{V_{t_{n+1}}^n} \times \{0\}$ in $U_{t_n}^n \times A \times B$ such that $\zeta(\Omega)$ is open in G_{n+1} and $\zeta|_{\Omega}$ is a C^∞-diffeomorphism onto $\zeta(\Omega)$. There exists $r > 0$ such that $U_{t_{n+1}}^n \times X_r^n \times V_{t_{n+1}}^n \times Y_r^n \subseteq \Omega$. Then

$$
\gamma_{n+1} \colon U_{t_{n+1}}^{n+1} = U_{t_{n+1}}^n \times X_{t_{n+1}}^n \to H_{n+1}\,, \quad \gamma_{n+1}(x, a) := \mu(\tfrac{r}{t_{n+1}}a)\gamma_n(x)
$$

and

$$
\eta_{n+1} \colon V_{t_{n+1}}^{n+1} = V_{t_{n+1}}^n \times Y_{t_{n+1}}^n \to G_{n+1}\,, \quad \eta_{n+1}(y, b) := \eta_n(y)\nu(y, \tfrac{r}{t_{n+1}}b)
$$

are smooth maps, and

$$
g_{n+1} \colon U_{t_{n+1}}^{n+1} \times V_{t_{n+1}}^{n+1} \to G_{n+1}\,, \quad g_{n+1}(x, y) := \gamma_{n+1}(x)\eta_{n+1}(y)
$$

is a C^∞-diffeomorphism onto an open subset of G_{n+1} (because so is $\zeta|_{\Omega}$). Consequently, γ_{n+1} is a C^∞-diffeomorphism onto an open subset of H_{n+1}. By construction, γ_{n+1} and η_{n+1} also have all other required properties. $\qquad\square$

3 Direct limit groups are topological groups with Lie algebra

Given a (Hausdorff) topological group G, let $C(\mathbb{R}, G)$ be the set of continuous G-valued functions on $\mathbb{R}$, and $\mathrm{Hom}_c(\mathbb{R}, G)$ be the subset of all continuous homomorphisms from $(\mathbb{R}, +)$ to G. We equip $C(\mathbb{R}, G)$ and $\mathrm{Hom}_c(\mathbb{R}, G)$ with the compact-open topology. Given $r \in \mathbb{R}$ and $\gamma \in \mathrm{Hom}_c(\mathbb{R}, G)$, we define $r\gamma \in \mathrm{Hom}_c(\mathbb{R}, G)$ via $(r\gamma)(t) := \gamma(tr)$ for $t \in \mathbb{R}$. Then

$$
\mathbb{R} \times \mathrm{Hom}_c(\mathbb{R}, G) \to \mathrm{Hom}_c(\mathbb{R}, G)\,, \quad (r, \gamma) \mapsto r\gamma \tag{8}
$$

is a continuous map [16, p. 111–112]. We recall from [16, Definition 2.11]:

Definition 3.1 G is called a *topological group with Lie algebra* if

(a) $(\gamma + \eta)(t) := \lim_{n\to\infty}(\gamma(t/n)\eta(t/n))^n$ exists for all $\gamma, \eta \in \mathrm{Hom}_c(\mathbb{R}, G)$, and maps $\gamma + \eta \colon \mathbb{R} \to G$ so obtained are continuous homomorphisms;

(b) $\mathrm{Hom}_c(\mathbb{R}, G)$ is a topological vector space with the addition defined in (a) and the scalar multiplication (8); and

(c) There exists a continuous bilinear map

$$[.,.]\colon \mathrm{Hom}_c(\mathbb{R}, G) \times \mathrm{Hom}_c(\mathbb{R}, G) \to \mathrm{Hom}_c(\mathbb{R}, G)$$

making it a Lie algebra, and such that

$$[\gamma, \eta](t^2) \;=\; \lim_{n\to\infty}(\gamma(t/n)\eta(t/n)\gamma(-t/n)\eta(-t/n))^{n^2},$$

for all $\gamma, \eta \in \mathrm{Hom}_c(\mathbb{R}, G)$ and $t \in \mathbb{R}$.

It is known that a Lie group G modelled on a locally convex space is a topological group with Lie algebra provided that G is *locally exponential* in the sense that G has a smooth exponential map $\exp_G \colon L(G) \to G$ which is a local diffeomorphism at 0 (see [21, Remark IV.1.22]). In particular, every Banach-Lie group and every finite-dimensional Lie group is a topological group with Lie algebra. Direct limits of finite-dimensional Lie groups always have a smooth exponential map, but they need not be locally exponential (see [5, Example 5.5]). Nonetheless, direct limit groups are topological groups with Lie algebra, as we verify now.

3.2 Recall that a Lie group G is said to *have an exponential map* if, for each $x \in L(G)$, there exists a (necessarily unique) smooth homomorphism $\gamma_x \colon \mathbb{R} \to G$ with derivative $\gamma_x'(0) = x$. In this case, define $\exp_G \colon L(G) \to G$, $\exp_G(x) := \gamma_x(1)$. Then $\gamma_x(t) = \exp_G(tx)$. As a first step towards our goal, consider a Lie group G with the following properties:

(a) G has an exponential map;

(b) Every continuous homomorphism $\mathbb{R} \to G$ is smooth;

(c) $(\gamma_x(t/n)\gamma_y(t/n))^n \to \gamma_{x+y}(t)$ as $n \to \infty$ and

$$(\gamma_x(t/n)\gamma_y(t/n)\gamma_x(-t/n)\gamma_y(-t/n))^{n^2} \to \gamma_{[x,y]}(t^2),$$

for all $x, y \in L(G)$ and $t \in \mathbb{R}$.

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Then the map
$$\Gamma_G \colon L(G) \to \mathrm{Hom}_c(\mathbb{R}, G), \quad x \mapsto \gamma_x$$
is a bijection. Furthermore, if we give $\mathrm{Hom}_c(\mathbb{R}, G)$ the unique Lie algebra structure which makes Γ_G an isomorphism of Lie algebras, then conditions (a) and (c) from Definition 3.1 are satisfied, and the scalar multiplication of the Lie algebra $\mathrm{Hom}_c(\mathbb{R}, G)$ is given by (8). Consequently, G *will be a topological group with Lie algebra if* Γ_G *is a homeomorphism.*

3.3 Now let $G_1 \subseteq G_2 \subseteq \cdots$ be finite-dimensional Lie groups, such that the inclusion maps $G_n \to G_{n+1}$ are smooth homomorphisms for all $n \in \mathbb{N}$. Let $G = \varinjlim G_n$ be the direct limit Lie group. Then (a) and (c) hold (see [7, Proposition 4.6 (a) and (b)]), and also (b) (cf. [7, Proposition 4.6 (c)]). Moreover, the map $\Gamma_{G_n} \colon L(G_n) \to \mathrm{Hom}_c(\mathbb{R}, G_n)$ is a homeomorphism (cf. [21, Remark IV.1.22]). Since the inclusion map $\mathrm{Hom}_c(\mathbb{R}, G_n) \to \mathrm{Hom}_c(\mathbb{R}, G)$ is continuous and $L(G) = \varinjlim L(G_n)$ as a topological space, Γ_G is continuous.

Theorem 3.4 *For each direct limit Lie group* $G = \varinjlim G_n$ *as in* **3.3**, *the mapping* $\Gamma_G \colon L(G) \to \mathrm{Hom}_c(\mathbb{R}, G)$ *is a homeomorphism and thus* G *is a topological group with Lie algebra.*

In view of the reduction steps performed in **3.2** and **3.3**, it only remains to show that Γ_G is an open map. This will follow from the next lemma.

3.5 It is useful to recall some standard facts first. Let G be a topological group. By definition, the sets
$$\lfloor K, U \rfloor_G := \{\gamma \in C(\mathbb{R}, G) \colon \gamma(K) \subseteq U\}$$
form a subbasis of the compact-open topology on $C(\mathbb{R}, G)$, for K ranging through the compact subsets of $\mathbb{R}$ and U through the open subsets of G. It is known that $C(\mathbb{R}, G)$ is a topological group under pointwise multiplication: $(\gamma \cdot \eta)(t) := \gamma(t)\eta(t)$. As a consequence,
$$\gamma \cdot \lfloor K, U \rfloor_G$$
is an open neighbourhood of γ for each $\gamma \in C(\mathbb{R}, G)$, compact set $K \subseteq \mathbb{R}$, and identity neighbourhood $U \subseteq G$. In fact, a basis of neighbourhoods of γ is obtained in this way. Consequently, if $\gamma \in \mathrm{Hom}_c(\mathbb{R}, G)$, then
$$(\gamma \cdot \lfloor K, U \rfloor_G) \cap \mathrm{Hom}_c(\mathbb{R}, G)$$
is an open neighbourhood of γ in $\mathrm{Hom}_c(\mathbb{R}, G)$, and the latter form a basis of neighbourhoods of γ.

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In the next lemma, G_1 and G_2 are finite-dimensional Lie groups such that $G_1 \subseteq G_2$ and the inclusion map $G_1 \to G_2$ is a smooth homomorphism. We identify $\mathfrak{g}_1 := L(G_1)$ with a Lie subalgebra of $\mathfrak{g}_2 := L(G_2)$ and assume that

$$\{x \in \mathfrak{g}_2 \colon \exp_{G_2}(\mathbb{R}x) \subseteq G_1\} = \mathfrak{g}_1 . \tag{9}$$

Lemma 3.6 *Let $x \in \mathfrak{g}_1$ and $U_j \subseteq \mathfrak{g}_j$ be a relatively compact, open neighbourhood of x for $j \in \{1, 2\}$, such that $U_1 \subseteq U_2$. Let $\varepsilon > 0$ and $J := [-\varepsilon, \varepsilon] \subseteq \mathbb{R}$. Let $V_1 \subseteq G_1$ be a closed identity neighbourhood such that*

$$(\gamma_x \cdot \lfloor J, V_1 \rfloor_{G_1}) \cap \mathrm{Hom}_c(\mathbb{R}, G_1) \subseteq \Gamma_{G_1}(U_1) \tag{10}$$

and $R_1 := \{y \in \mathfrak{g}_1 \colon \exp_{G_1}(Jy) \subseteq V_1\}$ is compact. Then there exists a closed identity neighbourhood $V_2 \subseteq G_2$ such that

$$(\gamma_x \cdot \lfloor J, V_2 \rfloor_{G_2}) \cap \mathrm{Hom}_c(\mathbb{R}, G_2) \subseteq \Gamma_{G_2}(U_2) , \tag{11}$$

and V_1 is contained in the interior V_2^0 of V_2 relative G_2. Furthermore, one can achieve that

$$R_2 := \{y \in \mathfrak{g}_2 \colon \exp_{G_2}(Jy) \subseteq V_2\}$$

is compact. If V_1 is compact, then also V_2 can be chosen compact.

Proof. Let $Q_1 \supseteq Q_2 \supseteq \cdots$ be a basis of compact identity neighbourhoods in G_2, and

$$W_m := V_1 Q_m \quad \text{for } m \in \mathbb{N}.$$

Then (11) holds with $V_2 := W_m$ for all sufficiently large $m \in \mathbb{N}$. In fact, otherwise we could find positive integers $m_1 < m_2 < \cdots$ and elements $\eta_n \in \lfloor J, W_{m_n} \rfloor$ for $n \in \mathbb{N}$ such that

$$\zeta_n := \gamma_x \cdot \eta_n \in \mathrm{Hom}_c(\mathbb{R}, G_2)$$

and $\zeta_n \notin \Gamma_{G_2}(U_2)$. Then $\zeta_n = \Gamma_{G_2}(z_n)$ with $z_n := \Gamma_{G_2}^{-1}(\zeta_n) \in \mathfrak{g}_2 \setminus U_2$.

Case 1: *If $(z_n)_{n \in \mathbb{N}}$ has a convergent subsequence*, then we may assume (after passage to the latter) that $z_n \to z$ as $n \to \infty$ for some $z \in \mathfrak{g}_2$. Define $\eta := \gamma_x^{-1} \cdot \gamma_z \in C(\mathbb{R}, G_2)$. Then

$$\eta(J) \subseteq V_1 . \tag{12}$$

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In fact, for each $t \in J$ and $n \in \mathbb{N}$, we have

$$
\begin{aligned}
\eta(t) \;=\; \gamma_x(t)^{-1} \exp_{G_2}(tz) \;&=\; \lim_{k\to\infty} \gamma_x(t)^{-1} \exp_{G_2}(tz_k) \\
\;&=\; \lim_{k\to\infty} \gamma_x(t)^{-1} \zeta_k(t) \;=\; \lim_{k\to\infty} \eta_k(t) \;\in\; W_{m_n}\,,
\end{aligned}
$$

because $\eta_k(t) \in W_{m_k} \subseteq W_{m_n}$ and W_{m_n} is closed. Hence

$$
\eta(t) \;\in\; \bigcap_{n\in\mathbb{N}} W_{m_n} \;=\; V_1\,,
$$

as asserted (where the last equality holds by [24, Lemma 3.17]).

As a consequence of (12), we have $\gamma_z(t) = \gamma_x(t)\eta(t) \in G_1$ for each $t \in J$ and hence $\exp_{G_2}(\mathbb{R}z) = \gamma_z(\mathbb{R}) \subseteq G_1$, whence $z \in \mathfrak{g}_1$, by (9). Hence $\gamma_z \in \mathrm{Hom}_c(\mathbb{R}, G_1)$ and thus $\eta \in C(\mathbb{R}, G_1)$. By (12), we have $\eta \in \lfloor J, V_1 \rfloor_{G_1}$. Hence

$$
\gamma_z = \gamma_x\eta \in (\gamma_x\lfloor J, V_1\rfloor_{G_1}) \cap \mathrm{Hom}_c(\mathbb{R}, G_1)\,,
$$

whence $\gamma_z \in \Gamma_{G_1}(U_1)$ (by (10)) and thus $z \in U_1$. But $z \in \mathfrak{g}_2 \setminus U_2$ and hence $z \notin U_1$, contradiction.

Case 2: *If $(z_n)_{n\in\mathbb{N}}$ has no convergent subsequence*, after passing to a subsequence we may assume that $\|z_n\| \to \infty$ as $n \to \infty$, where $\|.\|$ is a given norm on $\mathfrak{g}_2$. Since R_1 is compact, there exists $r > 0$ such that

$$
R_1 \;\subseteq\; \{y \in \mathfrak{g}_1 \colon \|y\| < r\} \;=:\; B_1\,. \tag{13}
$$

We let $B_2 := \{y \in \mathfrak{g}_2 \colon \|y\| < 2r\}$. As $2rz_n/\|z_n\|$ is contained in the compact set ∂B_2, after passage to a subsequence we may assume that $2rz_n/\|z_n\| \to b$ as $n \to \infty$ for some $b \in \partial B_2$. Abbreviate $r_n := 2r/\|z_n\|$. Then $r_n \to 0$, and we may assume that $r_n \leq 1$ for each n. For each $t \in J$, we then have

$$
\eta_n(tr_n) \;=\; \exp_{G_2}(-tr_nx)\exp_{G_2}(tr_nz_n) \;\to\; \exp_{G_2}(tb) \;\;\text{as}\;\; n \to \infty.
$$

Since $\eta_n(tr_n) \in W_{m_n}$, it follows that $\exp_{G_2}(tb) \in \bigcap_{n\in\mathbb{N}} W_{m_n} = V_1$. Hence $\exp_{G_2}(\mathbb{R}b) \subseteq G_1$ and thus $b \in \mathfrak{g}_1$. By the preceding, we have $b \in R_1$ and therefore $b \in B_1$, by (13). But $b \in \partial B_2$, whence $b \notin B_2$ and thus $b \notin B_1$, contradiction.

The final assertion is clear, since compactness of V_1 entails that also each W_m is compact. It only remains to show that R_2 can be chosen compact.

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However, applying the results already shown with 1, B_1 and B_2 instead of x, U_1 and U_2, respectively, we see that

$$\{y \in \mathfrak{g}_2 \colon \exp_{G_2}(Jy) \subseteq W_m\} \subseteq B_2$$

for all sufficiently large m. For such m, the set on the left is compact. $\quad\square$

Proof of Theorem 3.4. Since $G_0 = \varinjlim (G_n)_0$ (see **2.2**), $L(G) = L(G_0)$, $\mathrm{Hom}_c(\mathbb{R}, G) = \mathrm{Hom}_c(\mathbb{R}, G_0)$ and $\Gamma_G = \overrightarrow{\Gamma_{G_0}}$, after replacing G with G_0 and each G_n with $G_{n,0}$ we may assume that G and each G_n is connected. The discussion leading to (1) now shows that

$$L(G_n) \;=\; \{x \in L(G_{n+1}) \colon \exp_{G_{n+1}}(\mathbb{R}x) \subseteq G_n\}$$

(as needed in Lemma 3.6).

It remains to show that $\Gamma_G(U)$ is open in $\mathrm{Hom}_c(\mathbb{R}, G)$ for each open subset $U \subseteq L(G) = \bigcup_{n \in \mathbb{N}} L(G_n)$. We verify that $\Gamma_G(U)$ is a neighbourhood of $\Gamma_G(x)$ for each $x \in U$. After passage to a cofinal subsequence, we may assume that $x \in L(G_1)$. Then $U_n := L(G_n) \cap U$ is an open neighbourhood of x in $L(G_n)$ for each $n \in \mathbb{N}$, and $U_1 \subseteq U_2 \subseteq \cdots$. Since $\Gamma_{G_1}(U_1)$ is an open neighbourhood of γ_x in $\mathrm{Hom}_c(\mathbb{R}, G_1)$, there exists $\varepsilon > 0$ and an open identity neighbourhood $P \subseteq G_1$ such that

$$(\gamma_x \lfloor J, P \rfloor_{G_1}) \cap \mathrm{Hom}_c(\mathbb{R}, G_1) \;\subseteq\; \Gamma_{G_1}(U_1),$$

where $J := [-\varepsilon, \varepsilon]$ (cf. **3.5**). There exists a compact identity neighbourhood $V_1 \subseteq P$ such that

$$R_1 \;:=\; \{y \in L(G_1) \colon \exp_{G_1}(Jy) \subseteq V_1\}$$

is compact. Using now Lemma 3.6 and induction, we find compact identity neighbourhoods $V_n \subseteq G_n$ for all integers $n \geq 2$ such that

(i) $V_{n-1} \subseteq V_n^0$ (the interior relative G_n);

(ii) $(\gamma_x \cdot \lfloor J, V_n \rfloor_{G_n}) \cap \mathrm{Hom}_c(\mathbb{R}, G_n) \subseteq \Gamma_{G_n}(U_n)$, and

(iii) $R_n := \{y \in L(G_n) \colon \exp_{G_n}(Jy) \subseteq V_n\}$ is compact.

Since $Q_n := V_n^0$ is open in G_n and $Q_1 \subseteq Q_2 \subseteq \cdots$, the set $Q := \bigcup_{n \in \mathbb{N}} Q_n$ is open in G (see **2.3**). Then

$$(\gamma_x \cdot \lfloor J, Q \rfloor_G) \cap \mathrm{Hom}_c(\mathbb{R}, G)$$

is an open neighbourhood of γ_x in $\mathrm{Hom}_c(\mathbb{R}, G)$. The proof will be complete if we can show that

$$(\gamma_x \cdot \lfloor J, Q \rfloor_G) \cap \mathrm{Hom}_c(\mathbb{R}, G) \subseteq \Gamma_G(U). \tag{14}$$

To verify (14), let $\eta \in \lfloor J, Q \rfloor_G$ such that $\zeta := \gamma_x \cdot \eta \in \mathrm{Hom}_c(\mathbb{R}, G)$. Then $\zeta = \gamma_z$ with $z := \Gamma_G^{-1}(\zeta)$. There is $n \in \mathbb{N}$ such that $z \in L(G_n)$ and thus $\zeta \in \mathrm{Hom}_c(\mathbb{R}, G_n)$. Then $\eta = \gamma_x^{-1} \cdot \zeta \in C(\mathbb{R}, G_n)$. Since $\eta(J) \subseteq G_n$ is compact and $(Q_m \cap G_n)_{m \geq n}$ is an open cover of $\eta(J)$, there exists $m \geq n$ such that $\eta(J) \subseteq Q_m$. Thus $\eta \in \lfloor J, Q_m \rfloor_{G_m} \subseteq \lfloor J, V_m \rfloor_{G_m}$ and hence

$$\zeta = \gamma_x \cdot \eta \in (\gamma_x \cdot \lfloor J, V_m \rfloor_{G_m}) \cap \mathrm{Hom}_c(\mathbb{R}, G_m) \subseteq \Gamma_{G_m}(U_m),$$

using (ii). Then $\zeta \in \Gamma_{G_m}(U_m) \subseteq \Gamma_G(U)$ and thus (14) is established. $\quad\square$

If a Lie group is a topological group with Lie algebra, then this has useful consequences. We recall that it is an unsolved open problem (first formulated by John Milnor) whether every continuous homomorphism between infinite-dimensional Lie groups is smooth (as in the finite-dimensional case). The following result provides some positive information.

Proposition 3.7 *Let H be a locally exponential Lie group and G be a Lie group such that* (a)–(c) *from* **3.2** *are satisfied, $\Gamma_G \colon L(G) \to \mathrm{Hom}_c(\mathbb{R}, G)$ is a homeomorphism, and $\exp_G \colon L(G) \to G$ is smooth. Then every continuous homomorphism $\alpha \colon H \to G$ is smooth.*

Proof. The map $\beta := \mathrm{Hom}_c(\mathbb{R}, \alpha) \colon \mathrm{Hom}_c(\mathbb{R}, H) \to \mathrm{Hom}_c(\mathbb{R}, G)$, $\gamma \mapsto \alpha \circ \gamma$ is continuous. Since

$$
\begin{aligned}
\beta(\gamma + \eta)(t) &= \alpha \left(\lim_{n \to \infty} (\gamma(t/n)\eta(t/n))^n \right) \\
&= \lim_{n \to \infty} ((\alpha \circ \gamma)(t/n)(\alpha \circ \eta)(t/n))^n = (\beta(\gamma) + \beta(\eta))(t)
\end{aligned}
$$

for all $\gamma, \eta \in \mathrm{Hom}_c(\mathbb{R}, H)$, we see that β is a homomorphism of groups and hence a continuous linear map. Then $\theta := \Gamma_G^{-1} \circ \beta \circ \Gamma_H \colon L(H) \to L(G)$ is a continuous linear (and hence smooth) map such that $\exp_G \circ \theta = \alpha \circ \exp_H$. Since $\exp_G \circ \theta$ is smooth and $\exp_H$ is a local diffeomorphism, it follows that α is smooth on some open identity neighbourhood and hence smooth. $\quad\square$

4 Borel-type theorem for diffeomorphisms

This section is devoted to the proof of Borel's Theorem for Diffeomorphisms, as stated in the introduction.

As usual, $\mathrm{GL}_n(\mathbb{R})$ denotes the group of all invertible $n \times n$-matrices and $\mathrm{GL}(\mathbb{R}^n)$ the group of all automorphisms of the real vector space $\mathbb{R}^n$. The algebra of all linear endomorphism of $\mathbb{R}^n$ will be denoted by $\mathcal{L}(\mathbb{R}^n)$. If M is a finite-dimensional smooth manifold and $K \subseteq M$, we let $\mathrm{Diff}_K(M)$ be the Lie group of all C^∞-diffeomorphisms $\gamma \colon M \to M$ such that $\gamma(x) = x$ for all $x \in M \setminus K$ (see, e.g., [9]; cf. [18]). We write $C_K^\infty(M)$ for the space of smooth maps $\gamma \colon M \to \mathbb{R}$ such that $\gamma|_{M \setminus K} = 0$. Finally, we let $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be the space of $\mathbb{R}^n$-valued smooth maps on $\mathbb{R}^n$, equipped with the usual locally convex topology (the smooth compact-open topology).

Lemma 4.1 *Let $n \in \mathbb{N}$, $A \in \mathrm{GL}(\mathbb{R}^n)$ such that $\det(A) > 0$, and $K \subseteq \mathbb{R}^n$ be a compact 0-neighbourhood. Then there exists a smooth map*

$$\zeta \colon \mathbb{R} \to \mathrm{Diff}_K(\mathbb{R}^n)$$

with $\zeta(0) = \mathrm{id}_{\mathbb{R}^n}$, such that
$$\theta := \zeta(1)$$
satisfies $\theta|_U = A|_U$ for some 0-neighbourhood $U \subseteq K$.

Proof. Let R be a compact 0-neighbourhood contained in the interior K^0 of K. Then there is $\chi \in C_K^\infty(\mathbb{R}^n)$ such that $\chi|_R = 1$. It is well known that

$$\{B \in \mathrm{GL}(\mathbb{R}^n) \colon \det(B) > 0\}$$

is the connected component of $\mathrm{GL}(\mathbb{R}^n)$. We therefore find a smooth map $f \colon \mathbb{R} \to \mathrm{GL}(\mathbb{R}^n)$ with $f(0) = \mathrm{id}_{\mathbb{R}^n}$ and $f(1) = A$. Let $f'(t) \in \mathcal{L}(\mathbb{R}^n)$ be the derivative of f at $t \in \mathbb{R}$. We consider the time-dependent smooth vector field

$$v \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad v(t, x) := (f'(t) \circ f(t)^{-1})(x)$$

on $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$,

$$\gamma_x \colon \mathbb{R} \to \mathbb{R}^n, \quad \gamma_x(t) := f(t)(x)$$

is a smooth curve such that $\gamma_x(0) = x$ and

$$\gamma_x'(t) = f'(t)(x) = (f'(t) \circ f(t)^{-1})(f(t)(x)) = v(t, \gamma_x(t)).$$

Thus γ_x solves the initial value problem

$$y'(t) = v(t, y(t)), \quad y(0) = x.$$

Hence $\Phi_v \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $\Phi_{v,t}(x) := \gamma_x(t)$ is the flow of v (for fixed initial time $t_0 = 0$). Now consider

$$w \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad w(t, x) := \chi(x)v(t, x).$$

Then w admits a smooth flow $\Phi_w \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, and $\zeta(t) := \Phi_{w,t} \in \text{Diff}_K(\mathbb{R}^n)$ for each $t \in \mathbb{R}$. Since $\phi_{w,t}(0) = 0$ for each $t \in \mathbb{R}$ and the flow is continuous, we find a 0-neighbourhood $U \subseteq K$ such that

$$\Phi_{w,t}(x) \in R^0 \quad \text{for all } x \in U \text{ and } t \in [0, 1].$$

Since v and w coincide on $\mathbb{R} \times R^0$, it follows that $\Phi_{w,t}(x) = \Phi_{v,t}(x)$ for all $x \in U$ and $t \in [0, 1]$, entailing that

$$\Phi_{w,1}(x) = \Phi_{v,1}(x) = \gamma_x(1) = A(x) \quad \text{for all } x \in U.$$

By construction of the Lie group structure on $\text{Diff}_K(\mathbb{R}^n)$, the map

$$\kappa \colon \text{Diff}_K(\mathbb{R}^n) \to C_K^\infty(\mathbb{R}^n, \mathbb{R}^n), \quad \gamma \mapsto \gamma - \text{id}_{\mathbb{R}^n}$$

is a diffeomorphism onto an open subset of $C_K^\infty(\mathbb{R}^n, \mathbb{R}^n)$ (cf. [6]). Because

$$\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, x) \mapsto (\zeta(t) - \text{id}_{\mathbb{R}^n})(x) = \Phi_{w,t}(x) - x$$

is a smooth mapping, the exponential law for smooth mappings (see, e.g., [10, Lemma 12.1 (a)]) entails that $\kappa \circ \zeta$ (and hence also ζ) is smooth. Since $\zeta(0) = \text{id}_{\mathbb{R}^n}$ and $\zeta(1) = \Phi_{w,1}$, we see that ζ has all required properties. $\qquad\square$

Given $n \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$, let $e_j := (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbb{R}^n$ with 1 in the j-th slot.

Lemma 4.2 *Let $n \in \mathbb{N}$ and $a_\alpha \in \mathbb{R}^n$ for multi-indices $\alpha \in \mathbb{N}_0^n$ be given such that $A := (a_{e_1}, \ldots, a_{e_n}) \in \text{GL}_n(\mathbb{R})$ and $\det(A) > 0$. Then there exists $\phi \in \text{Diff}_c(\mathbb{R}^n)$ such that $\frac{(\partial^\alpha \phi)(0)}{\alpha!} = a_\alpha$ for each $\alpha \in \mathbb{N}_0^n$.*

Proof. After a translation, we may assume that $a_0 = 0$. Then the vectors a_α determine a formal diffeomorphism $f = \sum_\alpha a_\alpha x^\alpha$ of $\mathbb{R}^n$. Considering A as a formal diffeomorphism, the formal composition $g := A^{-1} \circ f$ is of the form $g = \sum_\alpha b_\alpha x^\alpha$ with vectors $b_\alpha \in \mathbb{R}^n$ such that $(b_{e_1}, \ldots, b_{e_n}) = \mathbf{1}$ is the identity matrix. Let θ be as in the previous lemma. If we can prove the lemma with the b_α in place of the a_α, leading to ϕ, then $\kappa := \theta \circ \phi$ is a diffeomorphism such that $\kappa|_W = A \circ \phi|_W$ for some 0-neighbourhood $W \subseteq \mathbb{R}^n$, and hence $\frac{(\partial^\alpha \kappa)(0)}{\alpha!} = a_\alpha$ for each $\alpha \in \mathbb{N}_0^n$. We may therefore assume now that $A = \mathbf{1}$.

Let $\|.\|$ be the maximum norm on $\mathbb{R}^n$. Then $\{x \in \mathbb{R}^n \colon \|x\| \leq \varepsilon\} \subseteq K$ for some $\varepsilon > 0$. Let $h \colon \mathbb{R} \to \mathbb{R}$ be a compactly supported smooth function such that $h(x) = x$ for x in some 0-neighbourhood, and $h(x) = 0$ if $|x| \geq \varepsilon$. Define $h_\alpha(x_1, \ldots, x_n) := h(x_1)^{\alpha_1} \cdots h(x_n)^{\alpha_n}$ and

$$M_{k,m} := \max_{|\beta|=m} \sum_{|\alpha|=k} \|a_\alpha\| \cdot \|\partial^\beta h_\alpha\|_\infty,$$

where $\|.\|_\infty$ is the supremum norm on the space $C_0(\mathbb{R})$ real-valued functions that vanish at infinity, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ the order of the multi-index $\alpha \in \mathbb{N}_0^n$. For each $k \in \mathbb{N}$ with $k \geq 2$, pick $c_k > 1$ such that

$$c_k^{m-k} M_{k,m} < \frac{2^{-k}}{n}$$

for all $m \in \mathbb{N}_0$ such that $m < k$. Define $\phi_k \colon \mathbb{R}^n \to \mathbb{R}^n$ via $\phi_k(x) := \sum_{|\alpha|=k} a_\alpha c_k^{-k} h_\alpha(c_k x)$. Then ϕ_k is smooth and

$$\partial^\beta \phi_k(x) = c_k^{|\beta|-k} \sum_{|\alpha|=k} a_\alpha (\partial^\beta h_\alpha)(c_k x),$$

whence

$$\|\partial^\beta \phi_k\|_\infty \leq c_k^{|\beta|-k} M_{k,|\beta|},$$

which is $\leq \frac{2^{-k}}{n}$ for all $k \in \mathbb{N}$ such that $k > |\beta|$. Hence the limit $\psi := \sum_{k=2}^\infty \phi_k$ exists in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, and clearly ψ has compact support. By the preceding, for $|\beta| = 1$ we have $\|\partial^\beta \phi_k\|_\infty \leq \frac{2^{-k}}{n}$ for each $k \geq 2$ and thus $\|\partial^\beta \psi\|_\infty \leq \sum_{k=2}^\infty \frac{2^{-k}}{n} = \frac{1}{2n}$, whence $\|\psi'(x)\|_{\mathrm{op}} \leq 1/2$ for each $x \in \mathbb{R}^n$ and thus $\sup\{\|\psi'(x)\|_{\mathrm{op}} \colon x \in \mathbb{R}^n\} \leq 1/2 < 1$. Therefore $\phi := \mathrm{id}_{\mathbb{R}^n} + \psi \in \mathrm{Diff}_c(\mathbb{R}^n)$ (see [6, Lemma 5.1]). By construction, $\frac{\partial^\alpha \phi(0)}{\alpha!} = a_\alpha$ for each α. $\qquad \square$

Proof of Borel's Theorem for Diffeomorphisms. Let p and a chart $\phi \colon M \supseteq U \to V \subseteq \mathbb{R}^n$ be as described in the theorem. After shrinking U, we

may assume that V is a euclidean ball with center 0. Composing now with a diffeomorphism $V \to \mathbb{R}^n$ which is the identity on some 0-neighbourhood and fixes 0, we can (and shall) assume instead that $V = \mathbb{R}^n$. Let $C \subseteq U$ be a compact neighbourhood of p and $K := \phi(C)$. Then the restriction map

$$\mathrm{Diff}_C(M) \to \mathrm{Diff}_C(U)$$

is an isomorphism of Lie groups and so is the map

$$\mathrm{Diff}_C(M) \to \mathrm{Diff}_K(\mathbb{R}^n)\,, \quad \gamma \mapsto \phi \circ \gamma \circ \phi^{-1}$$

(this is clear from the construction of the Lie group structure in [9]). It therefore suffices to consider the case where $M = \mathbb{R}^n$, $\phi = \mathrm{id}_{\mathbb{R}^n}$ and $p = 0$. In this case, the surjectivity of

$$T_p^\infty\colon\ \mathrm{Diff}_c(\mathbb{R}^n)_{p,0} \to \mathrm{Gf}_n(\mathbb{R})_0$$

has been established in Lemma 4.2. The Chain Rule for Taylor Polynomials entails that T_p^∞ is a homomorphism of groups. Since $\mathrm{Gf}_n(\mathbb{R})_0$ is an open subset of $\{\sum_\alpha a_\alpha x^\alpha \in \mathbb{R}[\![x_1, \ldots, x_n]\!] \colon a_0 = 0\} \cong \mathbb{R}^{\mathbb{N}_0^n \setminus \{0\}}$ with the product topology, and each component

$$(T_p^\infty)_\alpha\colon\ \mathrm{Diff}_c(\mathbb{R}^n)_{p,0} \to \mathbb{R}\,, \quad \gamma \mapsto \frac{(\partial^\alpha \gamma)(0)}{\alpha!}$$

is smooth, we deduce that T_p^∞ is smooth. This completes the proof. $\qquad\square$

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